JOURNAL OF DIFFERENTIAL EQUATIONS **149**, 191–210 (1998)
ARTICLE NO. DE983429

Global Attractor for the Cahn–Hilliard System with Fast Growing Nonlinearity*

Desheng Li and Chenkui Zhong

Department of Mathematics, Lanzhou University, Lanzhou 730000, People's Republic of China

Received September 23, 1996; revised December 29, 1997

1. INTRODUCTION

This paper is concerned with the study of the system of Cahn–Hilliard

$$u_t - \Delta K(u) = 0, \qquad \text{on} \quad R^+ \times \Omega, \tag{1.1}$$

$$K(u) = -\Gamma \Delta u + G(u), \qquad G(u) = \nabla_u \Phi(u), \tag{1.2}$$

where u: $R^+ \times \Omega \to R^N$, $\Gamma = [\Gamma_{ij}] \in R^{N \times N}$ is a symetric and positive definite matrix, Ω is a bounded domain in R^n $(n \leqslant 3)$ with sufficiently regular boundary $\partial\Omega$, $\Phi \in C^4(R^N; R)$. The system (1.1) is associated with the following homogeneous boundary conditions

$$\nabla_x u \vec{n} \,|_{x \in \partial\Omega} = \nabla_x (\Delta u)\, \vec{n} \,|_{x \in \partial\Omega} = 0 \tag{1.3}$$

(here $\nabla_x u = [\partial u_i / \partial x_j]$ is a gradient $N \times n$ matrix, $\vec{n}$ is the outward normal vector of $\partial\Omega$) and supplemented with an initial condition

$$u(0, x) = u_0(x), \qquad x \in \Omega. \tag{1.4}$$

For $N = 1$, the system reduces to the original Cahn–Hilliard model describing decomposition of a binary alloy which has been studied by many authors (for instance, see [1, 3, 4, 9, 11]). In general, it is proposed as a phase separation model in the case where the alloy consists of $N+1$ components. The derivation of (1.1)–(1.4) as well as many theoretical results concerning this problem can be found, for example, in [5, 6].

More recently, Cholewa and Dlotko [2] investigated the dynamical properties of (1.1)–(1.4). They proved that the system possesses global attractors on some affined subspaces of $[H^2(\Omega)]^N$ under the assumptions:

* Project (19671040) supported by NSFC.

0022-0396/98 $25.00

(1) There exists $M \in R$ such that

$$\Phi(u) \geqslant M, \qquad \forall u \in R^N;$$

(2) $u^T(\nabla_u \Phi(u)) \geqslant k_0 |u|^{2p} - k_1, (k_0, k_1 > 0)$;

(3) $|\partial^3 \Phi(u)/\partial u_i \partial u_j \partial u_k| \leqslant k_2(1 + |u|^{2p-3})$, $1 \leqslant i, j, k \leqslant N$, $u \in R^N$.

However, when the space dimension $n = 3$, the authors were forced to restrict their arguments to the case in which $p = 2$ because of technical difficulties which lie in estimating the $H^2(\Omega)$ norms of the solutions (By the way we also remark that even in case of $N = 1$ and Φ is a polynomial of order $2p$ as in (1.5), the exitence of global attractors in $H^2(\Omega)$ is still an open problem if the restriction on p is dropped, see [11, p. 385] and [4]. The main difficulty is the same as in the case in which $N > 1$). Due to the restriction on p, the known results given in [2], etc., may not apply to some important cases. For example, from the point of view of physics one of the most common candidates for Φ is a real polynomial of any even order $2p$ ($p \geqslant 1$) with all leading coefficients positive (see [3, 11], etc.),

$$\Phi(u) = \sum_{i=1}^{N} a_i u_i^{2p} + (a \text{ polynomial of order } < 2p). \tag{1.5}$$

Our goal in this work is to ensure, without any restriction on the growth of Φ, the existence of attractors for (1.1)–(1.4). Unlike those considered in the literature, our Φ may allow fast growing nonlinearity. Instead of (1)–(3), we impose on Φ the following structure conditions:

(C_1) There exists $\beta_1 > 0$ such that

$$\Phi(u) \geqslant -\beta_1, \qquad \forall u \in R^N;$$

(C_2) For $\forall u_0 \in R^N$, there exists $\beta_2 = \beta_2(u_0) > 0$ such that

$$u^T(\nabla_u \Phi(u + u_0)) \geqslant -\beta_2, \qquad \forall u \in R^N;$$

(C_3) There exists $\theta > 0$ such that

$$\liminf_{|u| \to \infty} \frac{u^T(\nabla_u \Phi(u)) - \theta \Phi(u)}{|u|^2} \geqslant 0;$$

(C_4) Φ is semiconvex, that is, there exists $\lambda > 0$ such that $\Phi(u) + (\lambda/2) |u|^2$ is convex.

Under the above assumptions, we prove that, the system (1.1)–(1.4) possesses global attractors on some affined subspaces of $[H^2(\Omega)]^N$ and $[H^3(\Omega)]^N$ given in Section 5. As a special case of $N = 1$, we have solved the open problem metioned above even more generally. Our method here differs significantly from that in [2, 4, 11]. Some of our important ideas come from [2, 3, 7, 9], etc.

Remark. 1.1. The condition (C_2) seems to be a little strange. However, there are many functions satisfying it. For example, in the case of $N=1$, if a function $\Phi(u)$ satisfies:

$$\Phi'(u)\,u \geqslant 0, \qquad \text{for} \quad u \in R, \quad |u| \geqslant M, \tag{1.6}$$

then Φ satisfies (C_2). In fact, for every $u_0 \in R$, there exists $M' = M'(u_0) > 0$ such that

$$|u+u_0| \geqslant M, \qquad \frac{u}{u+u_0} > 0, \qquad \text{for} \quad |u| \geqslant M'.$$

Therefore by (1.6),

$$\Phi'(u+u_0)\,u = \frac{u}{u+u_0}\,\Phi'(u+u_0)(u+u_0) \geqslant 0, \qquad \text{for} \quad |u| \geqslant M'.$$

Moreover, one can easily check that any polynomial as in (1.5) satisfies all the conditions (C_1)–(C_4).

Remark 1.2. Throughout this paper,the differential operators ∇, Δ are understood to be taken with respect to spatial variable x; otherwise we will always indecate the variable explicitly in the index, as in the case of $\nabla_u \Phi(u)$. For a matrix B, $\operatorname{tr} B$ is the trace of B, B^T is the transposed matrix of B.

2. MATHEMATICAL SETTING

We first introduce several notations which will be used throughout the following sections. We denote by H the space $L^2(\Omega)$, $(\cdot,\cdot)$ and $|\cdot|$ the scalar product and norm on H (and also on $H^N := [L^2(\Omega)]^N$ when there is no confusion). For each $v \in H$, $m(v)$ is the average of v,

$$m(v) = \frac{1}{|\Omega|}\int_\Omega v(x)\,dx.$$

$\dot{H}$ is the subspace of H,

$$\dot{H} = \{v \in H, m(v) = 0\}.$$

We introduce the unbounded linear operator A on H defined by $A=-\Delta$, with the domain

$$D(A)=\{v\in H^2(\Omega),\ \nabla v\cdot\vec{n}=0 \text{ on } \partial\Omega\}.$$

The operator A is positive, selfadjoint and possesses a basis of eigenvectors $\omega_j(j=0, 1, ...)$ which is orthonormal in H. It is associated with the eigenvalues $\lambda_j(j=0, 1, ...)$,

$$0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_j\to\infty.$$

The functions satisfy

$$\omega_0=|\Omega|^{-1/2}, \qquad \omega_j\in\dot{H}\ (\text{for } j\geqslant 1).$$

For $\forall s\in R$, we define A^s by setting

$$A^s v=\sum_{j=1}^{\infty}\lambda_j^s v_j\omega_j, \qquad \text{when} \quad v=\sum_{j=0}^{\infty} v_j\omega_j.$$

Note that

$$A^{-s}A^s v=A^s A^{-s}v=v-m(v). \tag{2.1}$$

For $s\in R$ we denote by V_s the space $[D(A^{s/2})]^N$, where $D(A^{s/2})$ is the domain of $A^{s/2}$,

$$D(A^{s/2})=\left\{v=\sum_{j=0}^{\infty} v_j\omega_j,\ \sum_{j=0}^{\infty}\lambda_j^s v_j^2<\infty\right\}.$$

V_s is endowed with the semiproduct and seminorm $(\cdot,\cdot)_s$ and $|\cdot|_s$,

$$(u, v)_s=\sum_{i=1}^{N}(A^{s/2}u_i, A^{s/2}v_i), \qquad |u|_s^2=(u, u)_s$$

and with the norm $\|\cdot\|_s$,

$$\|u\|_s=(|u|_s^2+|m(u)|^2)^{1/2},$$

where

$$m(u)=(m(u_i)), \qquad |m(u)|^2=\sum_{i=1}^{N} m(u_i)^2.$$

We have

$$\|u\|_0=(|u|_0^2+|m(u)|^2)^{1/2}=|u|.$$

When $s>0$, V_s is a subspace of $[H^s(\Omega)]^N$ and $\|\cdot\|_s$ is on V_s a norm equivalent to the usual one. Also, V_{-s} is the dual of V_s. Moreover, for $\forall s, s' \in R$, $s \leqslant s'$, we have the following Poincare-type inequality

$$|u|_s \leqslant \lambda_1^{-(s'-s)/2}\, |u|_{s'}, \qquad \forall u \in V_{s'} \tag{2.2}$$

and the interpolation inequality

$$|u|_{\sigma s_1+(1-\sigma)s_2} \leqslant |u|_{s_1}^{\sigma}\, |u|_{s_2}^{1-\sigma}, \qquad \sigma \in [0, 1]. \tag{2.3}$$

For general theory of spaces $H^s(\Omega)$, the interested reader is referred to [11] and also [10] by Rodriguez-Bernal.

3. SOME A PRIORI ESTIMATES

In this section we establish some a priori estimates for the solution u of (1.1)–(1.4). Some computations leading to our estimates are not reasonable because u may not be sufficiently regular. However, they can be justified by considering the Galerkin approximation u_k of u, which takes the form

$$u_k=(u_k^i)(1 \leqslant i \leqslant N), \qquad u_k^i=\sum_{j=0}^{k} g_{kj}^i(t)\, \omega_j,$$

where ω_j $(j=0, 1, 2, ...)$ the basis of eigenvectors of A given in Section 2. u_k satisfies:

$$(GAP) \begin{cases} \left(\dfrac{d}{dt} u_k + AK(u_k),\, v_j^i\right)=0, & 1 \leqslant i \leqslant N, \quad 0 \leqslant j \leqslant k, \\ u_k(0)=u_{0k}, \end{cases}$$

where u_{0k} is the orthogonal projection in H^N of u_0 onto H_k^N, H_k is the subspace of H spanned by $\omega_j (0 \leqslant j \leqslant k)$,

$$v_j^i=(\overbrace{0, ..., 0}^{i-1}, \omega_j, 0, ..., 0)^T.$$

Thanks to the classical theory on existence of solutions of ordinary differential equations, (GAP) possesses a unique solution u_k on some interval $[0, T_k)$. Since $\omega_j \in H^s(\Omega)$ $(s>0)$, we have $u_k \in C^1([0, T_k); V_s)$ for any $s>0$. Consider the equations in (GAP) corresponding to $j=0$. Recall that $\omega_0 := |\Omega|^{-1/2}$, one can easily deduce that the average $m(u_k)$ of u_k is conserved, or equivalently $g_{k0}^i(t) \equiv const$ for $1 \leqslant i \leqslant N$. Multiplying equation $((d/dt)\, u_k + AK(u_k), v_j^i)=0$ by $\lambda_j^{-1} g_{kj}^i$ $(j \geqslant 1)$ and summing all the relations for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant k$, we find that the energy-type equality (3.1) below holds with u_k, i.e.,

$$\frac{1}{2}\frac{d}{dt}|u_k|^2_{-1} + (K(u_k), \bar{u}_k) = 0,$$

where $\bar{u}_k = u_k - m(u_k)$. Similarly we can show that the other energy-type equalities of u in the proof of Lemmas 3.1–3.6 hold with u_k instead of u. Hence, as u_k is regular enough, all the computations in the following argument can be performed on u_k rigorously. Thus we conclude that the estimates given in Lemmas 3.1–3.6 hold if u therein are replaced by u_k (therefore one also understands that $T_k = \infty$). Moreover, we can see that all the estimates do not depend on k, which enables us to pass to the limit to find that u_k converges in suitable spaces with corresponding topology to the unique global solution u of (1.1)–(1.4) (consequently the global existence of the system is proved) (see Section 4). Because the estimates for u_k are k-independent, they remain valid naturally for u.

We now point out two basic facts about (1.1)–(1.4): It preserves the spatial average of any solution u in time, i.e.,

$$m(u(t)) \equiv m(u_0), \qquad \text{for} \quad t \geqslant 0;$$

moreover, it has a Lyapunov function $J(u)$ which decreases along any orbit $u(t)$ (this property is implicit in (3.16) below),

$$J(u) = \tfrac{1}{2}\int_\Omega tr(\nabla u^T \Gamma \nabla u)\, dx + \int_\Omega \Phi(u)\, dx.$$

For $\alpha > 0$, assume that $|m(u_0)| \leqslant \alpha$. In this section, we will always denote by u the solution of (1.1)–(1.4) corresponding to the initial value u_0, k_i the positive constants depending on Γ and Φ, $k_i(\alpha)$ the positive constants depending on α, Γ and Φ.

LEMMA 3.1. *For $\forall R$, $\alpha > 0$, there exist positive constants E_0, ρ_0 and t_0 such that*

(1) $|u|_{-1} \leqslant E_0$, $\forall t \geqslant 0$,

(2) $|u|_{-1} \leqslant \rho_0$, *for* $t \geqslant t_0$

when $|u_0|_{-1} \leqslant R$, $|m(u_0)| \leqslant \alpha$, where E_0, t_0 depend on R, α, Γ and Φ; ρ_0 only depends on α, Γ and Φ.

Proof. We multiply (1.1) by $A^{-1}u$ and integrate over Ω,

$$\frac{1}{2}\frac{d}{dt}|u|^2_{-1} + (K(u), \bar{u}) = 0. \tag{3.1}$$

where $\bar{u} = u - m(u) = u - m(u_0)$. Further, we have

$$(K(u), \bar{u}) = -(\Gamma\, \Delta u, \bar{u}) + (G(u), \bar{u})$$
$$= \int_\Omega tr(\nabla u^T \Gamma\, \nabla u)\, dx + (G(u), \bar{u}). \tag{3.2}$$

Since Γ is positive definite, we deduce that for some $\gamma_0 > 0$,

$$\int_\Omega tr(\nabla u^T \Gamma\, \nabla u)\, dx \geqslant \gamma_0\, |u|_1^2. \tag{3.3}$$

By (C_2), it can be easily seen that

$$\tfrac{1}{2}(G(u), u) - (G(u), m(u_0)) \geqslant -k_1(\alpha). \tag{3.4}$$

We also infer from (C_3) that for $\forall \varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(G(u), u) \geqslant \theta \int_\Omega \Phi(u)\, dx - \varepsilon\, |u|^2 - C_\varepsilon. \tag{3.5}$$

Collecting all of these inequalities together, we have

$$(K(u), \bar{u}) \geqslant \frac{1}{2} \int_\Omega tr(\nabla u^T \Gamma\, \nabla u)\, dx + \frac{\theta}{2} \int_\Omega \Phi(u)\, dx$$
$$+ \frac{\gamma_0}{2}\, |u|_1^2 - \varepsilon\, |u|^2 - C_\varepsilon - k_1(\alpha).$$

We choose ε sufficiently small such that

$$\frac{\gamma_0}{2}\, \|u\|_1^2 - \varepsilon\, |u|^2 \geqslant 0$$

and obtain

$$(K(u), \bar{u}) \geqslant \frac{1}{2} \int_\Omega tr(\nabla u^T \Gamma\, \nabla u)\, dx + \frac{\theta}{2} \int_\Omega \Phi(u)\, dx - k_2(\alpha). \tag{3.6}$$

If $\theta > 2$, then by (C_1) we have

$$(K(u), \bar{u}) = J(u) + \left(\frac{\theta}{2} - 1\right) \int_\Omega \Phi(u)\, dx - k_2(\alpha)$$
$$\geqslant J(u) - \left(\frac{\theta}{2} - 1\right) \beta_1\, |\Omega| - k_2(\alpha);$$

if $\theta \leqslant 2$, then

$$(K(u), \bar{u}) \geqslant \frac{\theta}{4} \int_{\Omega} tr(\nabla u^T \Gamma \nabla u)\, dx + \frac{\theta}{2} \int_{\Omega} \Phi(u)\, dx - k_2(\alpha)$$

$$= \frac{\theta}{2} J(u) - k_2(\alpha).$$

In general, we have

$$(K(u), \bar{u}) \geqslant k_1 J(u) - k_3(\alpha).$$

By (3.1), we find

$$\frac{d}{dt} |u|_{-1}^2 + k_1 J(u) \leqslant K_3(\alpha). \tag{3.7}$$

Moreover, by (3.1), (3.3), (3.6), and (C_1), we also have

$$\frac{d}{dt} |u|_{-1}^2 + \gamma_0 |u|_1^2 \leqslant k_4(\alpha).$$

By (2.2), $\lambda_1^2 |u|_{-1}^2 \leqslant |u|_1^2$, and thus

$$\frac{d}{dt} |u|_{-1}^2 + \gamma_0 \lambda_1^2 |u|_{-1}^2 \leqslant k_4(\alpha). \tag{3.8}$$

Thanks to the classical Gronwall Lemma, we derive from (3.8) that for $\forall t \geqslant 0$,

$$|u|_{-1}^2 \leqslant |u_0|_{-1}^2 \exp(-\gamma_0 \lambda_1^2 t) + \frac{k_4(\alpha)}{\gamma_0 \lambda_1^2} (1 - \exp(-\gamma_0 \lambda_1^2 t)). \tag{3.9}$$

Therefore

$$\limsup_{t \to \infty} |u|_{-1}^2 \leqslant \frac{k_4(\alpha)}{\gamma_0 \lambda_1^2}. \tag{3.10}$$

We assume that $|u_0|_{-1} \leqslant R$, $|m(u_0)| \leqslant \alpha$, $\rho_0 = (k_4(\alpha)/\gamma_0 \lambda_1^2)^{1/2} + 1$. By (3.9) and (3.10),

$$|u|_{-1}^2 \leqslant R^2 + \frac{k_4(\alpha)}{\gamma_0 \lambda_1^2} := E_0^2, \qquad \forall t \geqslant 0, \tag{3.11}$$

$$|u|_{-1} \leqslant \rho_0, \qquad \forall t \geqslant t_0. \tag{3.12}$$

LEMMA 3.2. *For* $\forall R, \alpha > 0$, *there exist* $E_2, \mu_2, \rho_2, t_2 > 0$ *such that*

(1) $|u|_2^2 \leqslant E_2(1 + 1/t^2)\exp(\mu_2 t), \forall t > 0;$

(2) $|u|_2 \leqslant \rho_2, \forall t \geqslant t_2,$

where E_2, μ_2 *and* t_2 *depends on* R, α, Γ *and* Φ; ρ_2 *depends only on* α, Γ *and* Φ.

Proof. Integrating (3.7) between t and $t+r$ $(r > 0)$, we find that

$$k_1 \int_t^{t+r} J(u)\, dt \leqslant k_3(\alpha)\, r + |u(t)|_{-1}^2 + |u(t+r)|_{-1}^2.$$

Since J decreases along orbit $u(t)$, we have

$$J(u(t+r)) \leqslant \frac{k_3(\alpha)}{k_1} + \frac{1}{k_1 r}\,(|u(t)|_{-1}^2 + |u(t+r)|_{-1}^2). \tag{3.13}$$

By (3.11)–(3.13), we deduce that

$$J(u) \leqslant \frac{k_3(\alpha)}{k_1} + \frac{2E_0^2}{k_1 r} := c_1(r), \qquad \forall r > 0, \quad t \geqslant r, \tag{3.14}$$

$$J(u) \leqslant \frac{k_3(\alpha)}{k_1} + \frac{2\rho_0^2}{k_1} := k_5(\alpha), \qquad \forall t \geqslant t_0 + 1. \tag{3.15}$$

We take the scalar product of (1.1) in H^N by $K(u)$ and obtain

$$\frac{d}{dt} J(u) + |K(u)|_1^2 \leqslant 0. \tag{3.16}$$

Integrate (3.16) between t and $t+s(s>0)$, we have

$$\int_t^{t+s} |K(u)|_1^2\, dt \leqslant J(u(t)) - J(u(t+s)).$$

By (C_1),

$$J(u) \geqslant \int_\Omega \Phi(u)\, dx \geqslant -\beta_1\, |\Omega|, \tag{3.17}$$

therefore by (3.14), (3.15),

$$\int_t^{t+s} |K(u)|_1^2\, dt \leqslant c_1(r) + \beta_1\, |\Omega| := c_2(r), \qquad \forall r > 0, \quad t \geqslant r, \tag{3.18}$$

$$\int_t^{t+s} |K(u)|_1^2\, dt \leqslant k_5(\alpha) + \beta_1\, |\Omega| := k_6(\alpha), \qquad \forall t \geqslant t_0 + 1. \tag{3.19}$$

We multiply (1.1) with $(d/dt)\,K(u)$ and integrate over Ω,

$$\left(u_t, \frac{d}{dt}\,K(u)\right)+\frac{1}{2}\frac{d}{dt}\,|K(u)|_1^2=0. \tag{3.20}$$

Further,

$$\begin{aligned}\left(u_t, \frac{d}{dt}\,K(u)\right)&=(u_t,\ -\Gamma\,\Delta u_t+Hess\ \Phi(u)\,u_t)\\ &=\int_\Omega tr((\nabla u_t)^T\,\Gamma\,\nabla u_t)\,dx+\int_\Omega u_t^T\,Hess\ \Phi(u)\,u_t\,dx\\ &\geqslant \gamma_0\,|u_t|_1^2+\int_\Omega u_t^T\,Hess\ \Phi(u)\,u_t\,dx,\end{aligned}$$

where $Hess\ \Phi(u)=(\partial^2\Phi(u)/\partial u_i\partial u_j))$. By (C_4),

$$\int_\Omega u_t^T\,Hess\ \Phi(u)\,u_t\,dx\geqslant -\lambda\,|u_t|^2,$$

hence

$$\frac{1}{2}\frac{d}{dt}\,|K(u)|_1^2+\gamma_0\,|u_t|_1^2\leqslant\lambda\,|u_t|^2.$$

Since $m(u_t)\equiv 0$, by (2.1),

$$\lambda\,|u_t|^2=\lambda\,|u_t|_0^2\leqslant\lambda\,|u_t|_{-1}\,|u_t|_1\leqslant\frac{\lambda^2}{4\gamma_0}\,|u_t|_{-1}^2+\gamma_0\,|u_t|_1^2,$$

therefore

$$\frac{d}{dt}\,|K(u)|_1^2\leqslant\frac{\lambda^2}{2\gamma_0}\,|u_t|_{-1}^2. \tag{3.21}$$

We also infer from Eq. (1.1) that

$$|u_t|_{-1}^2=|K(u)|_1^2,$$

thus

$$\frac{d}{dt}\,|K(u)|_1^2\leqslant\frac{\lambda^2}{2\gamma_0}\,|K(u)|_1^2. \tag{3.22}$$

Thanks to the Uniform Gronwall Lemma (see [11, Ch. III, Lemma 1.1]), we deduce immediately from (3.18), (3.19), and (3.22) that

$$|K(u)|_1^2 \leqslant \frac{c_2(r)}{s} \exp\left(\frac{\lambda^2}{2\gamma_0} s\right), \qquad \forall r, s > 0, \quad t \geqslant r + s, \tag{3.23}$$

$$|K(u)|_1^2 \leqslant k_6(\alpha) \exp\left(\frac{\lambda^2}{2\gamma_0}\right), \qquad \forall t \geqslant t_0 + 2. \tag{3.24}$$

Taking $r = s = \tau/2 = t/2$ in (3.23), we conclude that

$$|K(u)|_1^2 \leqslant c_1 \left(1 + \frac{1}{t^2}\right) \exp\left(\frac{\lambda^2}{4\gamma_0} t\right), \qquad \forall t > 0 \tag{3.25}$$

for some suitable constant c_1, c_1 depends on R, α, Γ and Φ.

We multiply $K(u)$ by Au and integrate over Ω, since Γ is positive definite, we have

$$\begin{aligned}(K(u), Au) &\geqslant \gamma_1 \, |u|_2^2 + (Au, G(u))\\ &= \gamma_1 \, |u|_2^2 + \int_\Omega \sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k}\right)^T \mathit{Hess}\, \Phi(u) \frac{\partial u}{\partial x_k} \, dx\\ &\geqslant (\text{by } (C_4)) \geqslant \gamma_1 \, |u|_2^2 - \lambda \, |u|_1^2,\\ \gamma_1 \, |u|_2^2 &\leqslant \lambda \, |u|_1^2 + (K(u), Au)\\ &\leqslant \left(\lambda + \frac{1}{2}\right) |u|_1^2 + \frac{1}{2} \, |K(u)|_1^2.\end{aligned} \tag{3.26}$$

By (3.3), (3.17),

$$\gamma_0 \, |u|_1^2 \leqslant \int_\Omega t_r(\nabla u^T \Gamma \, \nabla u) \, dx$$

$$= 2J(u) - 2 \int_\Omega \Phi(u) \, dx \leqslant 2\beta_1 \, |\Omega| + 2J(u),$$

therefore by (3.14), (3.15), we conclude that

$$|u|_1^2 \leqslant c_2 \left(1 + \frac{1}{t}\right), \qquad \forall t > 0, \tag{3.27}$$

$$|u|_1^2 \leqslant \rho_1^2, \qquad \forall t \geqslant t_0 + 1, \tag{3.28}$$

where $\rho_1 = (\gamma_0^{-1}(2k_5(\alpha) + 2\beta_1 \, |\Omega|))^{1/2}$, c_2 depends on R, α, Γ and Φ. Now we conclude the proof of Lemma 3.2 easily by collecting (3.24)–(3.28) together.

It can be seen that for some constant $\gamma_2 > 0$,

$$|u|_3 \leqslant \gamma_2 \, |\Gamma \, \Delta u|_1. \tag{3.29}$$

Therefore we have

$$|u|_3 \leqslant \gamma_2 \, |\Gamma \, \Delta u|_1 \leqslant \gamma_2 \, |K(u)|_1 + \gamma_2 \, |G(u)|_1. \tag{3.30}$$

Since $H^2(\Omega) \subset L^\infty(\Omega)$ (for $n \leqslant 3$), we find

$$|G(u)|_1 \leqslant N \max_{\substack{|v| \leqslant |u|_{L^\infty} \\ 1 \leqslant i, \, j \leqslant N}} \frac{\partial^2 \Phi(v)}{\partial v_i \, \partial v_j} \, |u|_1. \tag{3.31}$$

By (3.27)–(3.31) and Lemma 3.2, we deduce immediately the following Lemma.

LEMMA 3.3. *For $\forall R$, $\alpha > 0$, there exist a positive continuous function $\mu(t)$ defined on $(0, \infty)$ and two positive constants ρ_3, t_3 such that when $|u_0|_{-1} \leqslant R$, $|m(u_0)| \leqslant \alpha$,*

(1) $|u|_3 \leqslant \mu(t)$, $\forall t > 0$;

(2) $|u|_3 \leqslant \rho_3$, *for* $t \geqslant t_3$,

where $\mu(t)$ and t_3 depend on R, α, Γ and Φ; ρ_3 depends on α and Γ, Φ.

LEMMA 3.4. *For $\forall R > 0$, there exists a time T_0 depending on R, α, Γ, and Φ such that*

$$\|u(t)\|_2^2 \leqslant 2(1 + R^2), \qquad \forall t \in [0, T_0] \tag{3.32}$$

when $\|u_0\|_2 \leqslant R$.

Proof. We multiply (1.1) with $\Delta^2 u$ and integrate over Ω,

$$\frac{1}{2} \frac{d}{dt} \, |u|_2^2 + (\Gamma \, \Delta^2 u, \, \Delta^2 u) = (\Delta G(u), \, \Delta^2 u).$$

It is easily seen that for some $\gamma_3 > 0$,

$$(\Gamma \, \Delta^2 u, \, \Delta^2 u) \geqslant \gamma_3 \, |u|_4^2, \tag{3.33}$$

therefore

$$\frac{1}{2}\frac{d}{dt}|u|_2^2+\frac{1}{2}(\Gamma\,\Delta^2 u,\Delta^2 u)+\frac{1}{2}\gamma_3\,|u|_4^2$$

$$\leqslant(\Delta G(u),\Delta^2 u)\leqslant\frac{1}{2\gamma_3}|\Delta G(u)|^2+\frac{\gamma_3}{2}|u|_4^2,$$

$$\frac{d}{dt}|u|_2^2+(\Gamma\,\Delta^2 u,\Delta^2 u)\leqslant\frac{1}{\gamma_3}|\Delta G(u)|^2. \tag{3.34}$$

We suppose that u is sufficiently regular (this is right for the Galerkin approximation u_k of u), therefore

$$\|u(t)\|_2^2\leqslant 2(1+R^2),\qquad \forall t\in[0,T_0]$$

for some $T_0>0$ to be determined. Since $H^2(\Omega)\subset L^\infty(\Omega)$, there exists $c_3=c_3(R)$ such that

$$|u(t)|_{L^\infty}\leqslant c_3,\qquad \text{for}\quad t\in[0,T_0].$$

Futher, we have

$$\Delta G(u)=(g_k),\qquad g_k=\sum_{l=1}^{n}\left(\sum_{i,j=1}^{N}\frac{\partial^2 G_k}{\partial u_i\,\partial u_j}\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}+\sum_{i=1}^{N}\frac{\partial G_k}{\partial u_i}\frac{\partial^2 u_i}{\partial x_l^2}\right), \tag{3.35}$$

where $G_k=\partial\Phi/\partial u_k$, $k=1,2,\ldots,N$. Let

$$M_0=\max_{\substack{v\in R^N,\,|v|\leqslant c_3\\ |a|\leqslant 3}}|D^a\Phi(v)|,$$

here $D^a\Phi(v)$ denote the derivative of Φ of order $a\in Z_+^N$. Then (by Cauchy inequality)

$$|g_k|^2\leqslant M_0^2\int_\Omega\left(\sum_{l=1}^{n}\left(\sum_{i,j=1}^{N}\left|\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}\right|+\sum_{i=1}^{N}\left|\frac{\partial^2 u_i}{\partial x_l^2}\right|\right)\right)^2 dx$$

$$\leqslant M_1^2\int_\Omega\sum_{l=1}^{n}\left(\sum_{i,j=1}^{N}\left|\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}\right|^2+\sum_{i=1}^{N}\left|\frac{\partial^2 u_i}{\partial x_l^2}\right|^2\right)dx$$

$$=M_1^2\,|u|_2^2+M_1^2\int_\Omega\sum_{l=1}^{n}\sum_{i,j=1}^{N}\left|\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}\right|^2 dx$$

$$\leqslant M_1^2\,|u|_2^2+N^2M_1^2\int_\Omega\sum_{l=1}^{n}\sum_{i=1}^{N}\left|\frac{\partial u_i}{\partial x_l}\right|^4 dx,$$

where $M_1 = nN(N+1)M_0$. Since $H^1(\Omega) \subset L^4(\Omega)$, we find that

$$\int_\Omega \sum_{l=1}^n \sum_{i=1}^N \left| \frac{\partial u_i}{\partial x_l} \right|^4 dx \leqslant c_1' \|u\|_2^4$$

and thus

$$\begin{aligned} |\Delta G(u)|^2 = \sum_{k=1}^N |g_k|^2 &\leqslant NM_1^2(1 + c_1' N^2 \|u\|_2^2) \|u\|_2^2 \\ &\leqslant NM_1^2 c_2'(1 + \|u\|_2^2) \|u\|_2^2 \\ &\leqslant c_4(1 + \|u\|_2^2), \qquad \text{for} \quad t \in [0, T_0], \end{aligned}$$

where $c_4 = 2(1+R^2)\, NM_1^2 c_2'$, $c_2' = \max(1, c_1' N^2)$. Since $m(u) \equiv m(u_0)$, by (3.34),

$$\begin{aligned} \frac{d}{dt}(1 + \|u\|_2^2) &\leqslant \frac{c_4}{\gamma_3}(1 + \|u\|_2^2), \qquad t \in [0, T_0], \\ 1 + \|u\|_2^2 &\leqslant (1 + R^2) \exp\left(\frac{c_4}{\gamma_3} t\right), \qquad t \in [0, T_0]. \end{aligned} \tag{3.36}$$

We deduce from (3.36) that (3.32) remains valid at least for

$$t \leqslant T_0 := \frac{\gamma_3}{c_4} \log 2.$$

Lemmas 3.2 and 3.4 give a global estimate of $\|u\|_2$ on $[0, \infty)$. As a simple consequence, we have

LEMMA 3.5. *Let $u_0 \in V_2$. Then for $\forall R > 0$, there exists a constant $C_T > 0$ depending on R, Γ, Φ, and T such that*

$$\|u\|_{L^2(0, T; V_4)} \leqslant C_T$$

when $\|u_0\|_2 \leqslant R$.

Proof. This estimate is obtained by a simple integration of (3.34) between 0 and T.

LEMMA 3.6. *For $\forall R, \alpha > 0$, there exists a constant $E_3 > 0$ depending on R, α, Γ, and Φ such that*

$$\|u\|_3 \leqslant E_3, \qquad \textit{for} \quad t \in [0, T_0]$$

when $\|u_0\|_3 \leqslant R$.

Proof. We multiply (1.1) by Au_t to obtain

$$|u_t|_1^2+\frac{1}{2}\frac{d}{dt}\int_\Omega t_r((\nabla \Delta u)^T\,\Gamma\,\nabla\Delta u)\,dx$$

$$=(\Delta G(u),\,-\Delta u_t)\leqslant\frac{1}{2}\,|\nabla\Delta G(u)|^2+\frac{1}{2}\,|u_t|_1^2,$$

$$\frac{d}{dt}\int_\Omega t_r((\nabla\Delta u)^T\,\Gamma\,\nabla\Delta u)\,dx\leqslant|\nabla\Delta G(u)|^2. \tag{3.37}$$

By (3.35),

$$\frac{\partial}{\partial x_\theta}\,\Delta G_k(u)=\sum_{l=1}^n\left(\sum_{i,\,j,\,m=1}^N\frac{\partial^3 G_k}{\partial u_i\partial u_j\partial u_m}\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}\frac{\partial u_m}{\partial x_\theta}\right.$$
$$+2\sum_{i,\,j=1}^N\frac{\partial^2 G_k}{\partial u_i\partial u_j}\frac{\partial^2 u_i}{\partial x_l\partial x_\theta}\frac{\partial u_j}{\partial x_l}+\sum_{i,\,j=1}^N\frac{\partial^2 G_k}{\partial u_i\partial u_j}\frac{\partial u_j}{\partial x_\theta}\frac{\partial^2 u_i}{\partial x_l^2}$$
$$\left.+\sum_{i=1}^n\frac{\partial G_k}{\partial u_i}\frac{\partial^3 u_i}{\partial x_l^2\,\partial x_\theta}\right),$$

where $G_k(u)=\partial\Phi(u)/\partial u_k$. Since $H^2(\Omega)\subset L^\infty(\Omega)$, by Lemma 3.4, there exists $c_5=c_5(R)>0$ such that

$$|u|_{L^\infty}\leqslant c_5,\qquad\forall t\in[0,\,T_0].$$

Therefore

$$\left|\frac{\partial}{\partial x_\theta}\,\Delta G_k(u)\right|\leqslant M_2\sum_{i=1}^n\left(\sum_{i,\,j,\,m=1}^N\left|\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}\frac{\partial u_m}{\partial x_\theta}\right|+2\sum_{i,\,j=1}^N\left|\frac{\partial^2 u_i}{\partial x_l\partial x_\theta}\frac{\partial u_j}{\partial x_l}\right|\right.$$
$$\left.+\sum_{i,\,j=1}^N\left|\frac{\partial u_j}{\partial x_\theta}\frac{\partial^2 u_i}{\partial x_l^2}\right|+\sum_{i=1}^n\left|\frac{\partial^3 u_i}{\partial x_i^2\,\partial x_\theta}\right|\right),$$

where

$$M_2=\max_{\substack{v\in R^N,\ |v|\leqslant c_5\\ |a|\leqslant 4}}|D^a\Phi(v)|.$$

Futher, we have (since $H^1(\Omega) \subset L^6(\Omega)$),

$$\left| \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l} \frac{\partial u_m}{\partial x_\theta} \right| \leqslant |\nabla u|_{L^6}^3 \leqslant c_3' \, \|u\|_2^3 \leqslant c_6, \qquad \text{for} \quad t \in [0, T_0];$$

$$\left| \frac{\partial^2 u_i}{\partial x_l \partial x_\theta} \frac{\partial u_j}{\partial x_l} \right| \leqslant |\nabla u|_{L^\infty} \, |u|_2$$

$$\leqslant c_7 \, |\nabla u|_{L^\infty} \leqslant c_8 \, \|u\|_3, \qquad t \in [0, T_0];$$

$$\left| \frac{\partial u_j}{\partial x_\theta} \frac{\partial^2 u_i}{\partial x_l^2} \right| \leqslant |\nabla u|_{L^\infty} \, |u|^2 \leqslant c_8 \, \|u\|_3, \qquad t \in [0, T_0].$$

Collecting all these inequalities above together, we find

$$\left| \frac{\partial}{\partial x_\theta} \Delta G_k(u) \right|^2 \leqslant c_9(1 + \|u\|_3^2), \qquad \forall t \in [0, T_0].$$

By (3.37),

$$\frac{d}{dt} \int_\Omega t_r((\nabla \Delta u)^T \, \Gamma \, \nabla \Delta u) \, dx \leqslant nNc_9(1 + \|u\|_3^2).$$

By (3.3),

$$\|u\|_3^2 \leqslant \frac{1}{\gamma_0} \int_\Omega t_r((\nabla \Delta u)^T \, \Gamma \, \nabla \Delta u) \, dx + |m(u_0)|^2, \tag{3.38}$$

and hence

$$\frac{d}{dt} \int_\Omega t_r((\nabla \Delta u)^T \, \Gamma \, \nabla \Delta u) \, dx$$

$$\leqslant c_{10} \left(1 + \int_\Omega t_r((\nabla \Delta u)^T \, \Gamma \, \nabla \Delta u) \, dx \right), \qquad t \in [0, T_0]. \tag{3.39}$$

(3.38) and (3.39) imply the estimate of $\|u\|_3$ on the interval $[0, T_0]$. The proof is complete.

4. GLOBAL EXISTENCE AND UNIQUENESS

THEOREM 4.1. *Assume that Φ satisfies* (C_1)–(C_4), *then for* $\forall u_0 \in V_s$ ($s = 2, 3$), *the system* (1.1)–(1.4) *possesses a unique solution u satisfying*

(1) $u \in C([0, \infty); V_s) \cap L^\infty(0, \infty; V_s)$;

(2) $u \in L^2(0, T; V_4)$, $\forall T > 0$.

Proof. As it is stated in Section 3, all the estimates given in Lemmas 3.1–3.6 hold for the approximate solutions u_k of (1.1)–(1.4), therefore the proof of existence is standard by the Galerkin method, we only need to point out the following fact which is used in the argument:

If $\{u_k\}$ is bounded in $L^\infty(0, T; V_2)$, $u_k \to u$ in $L^2(0, T; V_2)$ weakly and in $L^\infty(0, T; V_0)$ weak-star, $(d/dt)\, u_k \to (d/dt) u$ in $L^2(0, T; V'_2)$ weakly, then

$$G(u_k) \to G(u) \text{ strongly in } L^2(0, T; V_0). \tag{4.1}$$

We prove that any subsequence of $\{u_k\}$ possesses a subsequence which satisfies (4.1), therefore $\{u_k\}$ satisfies (4.1). For this purpose, it suffices to show that $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ satisfying (4.1).

Indeed, since $H^2(\Omega) \subset L^\infty(\Omega)$ (for $n \leqslant 3$), we have for some constant $C_T > 0$,

$$|u_k|_{L^\infty} \leqslant C_T, \qquad \forall t \in [0, T], \quad k \geqslant 1. \tag{4.2}$$

Due to a compactness theorem (see [12, Ch. III, Section 2]),

$$u_k \to u \qquad \text{in} \quad L^2(0, T; V_0).$$

Therefore for some subsequence $\{u_{k_i}\}$,

$$u_{k_i}(t, x) \to u(t, x), \qquad \text{for a.e.} \quad (t, x) \in [0, T] \times \Omega,$$

and thus

$$G(u_{k_i}(t, x)) \to G(u(t, x)), \qquad \text{for a.e.} \quad (t, x) \in [0, T] \times \Omega. \tag{4.3}$$

By (4.2) and the classical Lebesgue Control Covergence Theorem, we deduce immediately that $\{u_{ki}\}$ satisfies (4.1).

The argument on the continuity of $u(t)$ in time with values in V_s is also classical, we omit it, the reader is referred to [11, Ch. III] for such aspects.

We now prove the uniqueness results.

Let $u_0, v_0 \in V_2$, u and v are the solutions of (1.1)–(1.4) corresponding to the initial values u_0 and v_0. Let $w = u - v$. Then w satisfies

$$w_t + \Gamma \Delta^2 w = \Delta(G(u) - G(v)). \tag{4.4}$$

We multiply (4.4) with $\Delta^2 w$ and obtain (by (3.33)).

$$\frac{1}{2}\frac{d}{dt}|w|_2^2+\gamma_3|w|_4^2 \leqslant (\Delta(G(u)-G(v)), \Delta^2 w)$$

$$\leqslant \frac{1}{4\gamma_3}|\Delta(G(u)-G(v))|^2+\gamma_3|w|_4^2,$$

$$\frac{d}{dt}|w|_2^2 \leqslant \frac{1}{2\gamma_3}|\Delta(G(u)-G(v))|^2. \tag{4.5}$$

By an anologous argument as in (14)–(16) of [8], we can prove that when $\|u_0\|_2, \|v_0\|_2 \leqslant R$,

$$|\Delta(G(u)-G(v))| \leqslant C\,\|w\|_2, \tag{4.6}$$

Where C depends on R, Γ, Φ, and T. Therefore (since $m(w) \equiv const$)

$$\frac{d}{dt}\|w\|_2^2 \leqslant C\,\|w\|_2^2, \qquad \text{for} \quad t \in [0, T].$$

We deduce immediately that

$$\|u(t)-v(t)\|_2 \leqslant C\,\|u_0-v_0\|_2, \qquad \forall t \in [0, T]. \tag{4.7}$$

The uniqueness result in V_3 can be proved in a quite similar manner (in this case we multiply (4.4) with Aw_t), we omit it.

5. EXISTENCE OF GLOBAL ATTRACTORS

Theorem 4.1 enables us to define a semigroup $\{s(t)\}_{t \geqslant 0}$ on V_s $(s=2, 3)$,

$$S(t)u_0 = u(t), \qquad \forall u_0 \in V_s, \quad t \geqslant 0,$$

where $u(t)$ is the solution of (1.1)–(1.4) corresponding to initial value u_0.

Since the system preserves the spatial average of solutions, it is impossible to construct a compact global attractor for $\{S(t)\}$ on the whole space V_s. However, we find that the restriction of $\{S(t)\}$ on the affined space

$$V_s(a) = \{u \in V_s, |m(u)| \leqslant a\} \qquad (a>0)$$

is a well defined semigroup.

We infer from Theorem 4.1 and its proof that $\{S(t)\}$ is strongly continuous. Furthermore, $\{S(t)\}$ takes bounded sets of V_s into bounded sets. Using the general results of Hale [7] and the idea of a sectorial operator [8] ($\Gamma\Delta^2$ is a sectorial operator), we can now formulate the following theorem. Its proof is an anolog of that of Theorem [4] 1 and Theorem [2] 1, we omit it.

THEOREM 5.1. *The semigroup* $\{S(t)\}$ *generated by the Cahn–Hilliard system* (1.1)–(1.4) *possesses on* $V_s(a)$ $(s=2, 3)$ *a global attractor* $A(a)$ *which attracts each point of* $V_s(a)$. *Furthermore,* $A(a)$ *is connected and bounded in* V_s.

Remark 5.1. One can easily check that $A(a)$ does not depend on s. Moreover, we remark that $A(a)$ as an attractor of $\{S(t)\}$ on $V_2(a)$ can also be obtained by using the general existence result of attractors in [11, Ch. I]. Thus we also deduce that $A(a)$ attracts any bounded set of $V_2(a)$ in the topology of V_2.

Remark 5.2. Since for two solutions u, v of (1.1)–(1.4), the Lipshcitz condition (4.7) holds, based on the abstract backward uniqueness result stated in [11, p. 170], it follows that the restriction of $\{S(t)\}$ on $A(a)$ is a one-to-one mapping for each $t \geqslant 0$ fixed. Thus the semigroup $\{S(t)\}$ restricted on $A(a)$ can be extended to a group.

Remark 5.3. All our results remain valid if the Neumann boundary condition (1.3) is replaced by the periodic condition

$$D^a u\,|_{x_i=0} = D^a u\,|_{x_i=L_i}, \qquad i=1, ..., n,$$

when $\Omega = \prod_{i=1}^n (0, L_i)$, where $D^a u$ denotes any derivative of u of order less than or equal to 3.

ACKNOWLEDGEMENTS

The authors express their gratitude to Professors Chen Wenyuan and Fan Xianling for their guidence and encouragement, also to an anonymous referee for a number of valuable comments and suggestions.

REFERENCES

1. N. Alikakos, P. W. Bates, and G. Fusco, Slow motion for the Cahn–Hilliard equation in one space dimension, *J. Differential Equations* **90** (1991), 81–135.
2. J. W. Cholewa and T. Dlotko, Global attractor of the Cahn–Hilliard system, *Bull. Austral. Math. Soc.* **49** (1994), 277–202.
3. A. Debussche and L. Dettori, On the Cahn–Hilliard equation with a logarithmic free energy, *Nonlinear Anal.* **24** (1995), 1491–1514.
4. T. Dlotko, Global attractor for the Cahn–Hilliard equation in H^2 and H^3, *J. Differential Equations* **113** (1994), 381–393.
5. C. M. Elliot and S. Luckhaus, A generalized diffusion equation for phase seperation of a multi-component mixture with interfacial free energy, IMA, Preprint 887 (1991).
6. D. J. Eyre, System of Cahn–Hilliard equations, University of Minnesota, AHPCRC Preprint 92-102 (1992).

7. J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Amer. Math. Soc., Providence, RI, 1988.
8. D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Springer-Verlag, Berlin/Heidelberg/New York, 1981.
9. B. Nicolaenko, B. Sheurer, and R. Temam, Some global dynamical properties of a class of pattern formation equations, *Comm. Partial Differential Equations* **14** (1989), 245–297.
10. A. Rodriguez-Bernal, CDSNS Report 61 (1991), Georgia Tech., Atlanta, GA.
11. R. Temam, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, Berlin/New York, 1988.
12. R. Temam, "Navier–Stokes Equations, Theory and Numerical Analysis," 3rd rev. ed., North-Holland, Amsterdam, 1984.